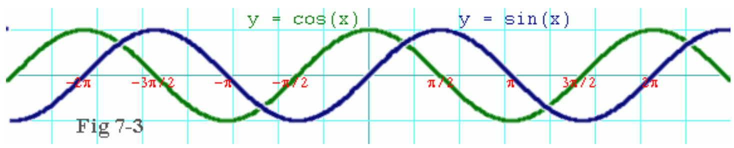
Common Trig Identities and their Derivations



Here again is figure 7-3, so you can look at the plots of $\sin(x)$ and $\cos(x)$ while we discuss them. And here is a table of values of $\sin(x)$ and $\cos(x)$ for some common values of x. These results can all be derived from basic geometry.

x	degrees	$\sin(x)$	$\cos(x)$	x	degrees	$\sin(x)$	$\cos(x)$
0	$0°$	0	1	π	$180°$	0	-1
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{7\pi}{6}$	$210°$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$	$225°$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{4\pi}{3}$	$240°$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{\pi}{2}$	$90°$	1	0	$\frac{3\pi}{2}$	$270°$	-1	0
$\frac{2\pi}{3}$	$120°$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{5\pi}{3}$	$300°$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{3\pi}{4}$	$135°$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{7\pi}{4}$	$315°$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$150°$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{11\pi}{6}$	$330°$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
				2π	$360°$	0	1

Table 7-1: Values of $\sin(x)$ and $\cos(x)$

I'd like you to think for a moment about the train on the circular track again. Suppose that train is at the station on Main Street and 10th Avenue East. What is different about when it backs up from when it moves forward? If it backs up 50 meters, doesn't it end up just as far east of Main Avenue as it does going forward 50 meters? Isn't the same true if it backs up any amount versus going forward by that same amount? Now recall that how far east or west of Main Avenue the train is is how we introduced cosine. So what this illustrates is a symmetry of the cosine functions:

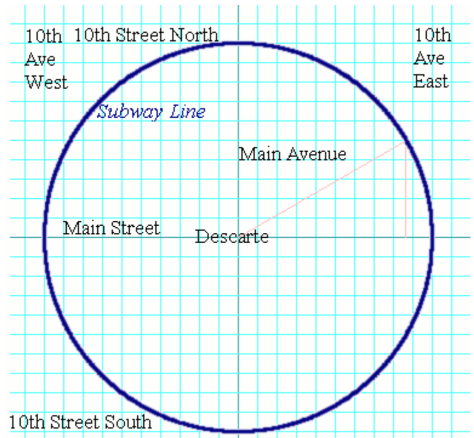


Fig. 7-1 New Atlantis Capital City

$$\cos(-x) = \cos(x) \tag{7.1b-1}$$

In math lingo, we say that cosine is an *even* function. Any $f(x)$ that obeys the property, $f(-x) = f(x)$ is said to be an *even* function. Note, for example, that x raised to any even power is an even function. That, by the way, is the origin of the calling such functions even.

Similarly, what happens differently to the train's north-south position depending upon whether it goes backward 50 meters or forward 50 meters? Well this time, there is a difference. But there is also a symmetry here as well. The train will end up just as far *north* of Main Street by going 50 meters forward as it would end up *south* of Main Street by by going 50 meters backward. And again you could substitute any other amount for 50 meters, and the same would still be true. Now recall that how far north or south of Main Street the train is is how we introduced sine. So with sine, the relationship is

$$\sin(-x) = -\sin(x) \tag{7.1b-2}$$

As you might have guessed, in math lingo we say that sine is an *odd* function. Any $f(x)$ that obeys $f(-x) = -f(x)$ is said to be an *odd* function. And as you'd expect, we get that terminology because x raised to any odd power is an odd function.

(*Food for thought:* Can you show that the only function that is simultaneously an even function and an odd function is $f(x) = 0$? Can you also show that every real function of a real variable is the sum of an even function and an odd function? Think about the sum and the difference of $f(x)$ and $f(-x)$.)

Here is another property of sine and cosine that should be evident from the circular track model and the plot shown in figure 7-3. Both sine and cosine have a *period* of 2π. That is, wherever you are on the track, if you go 2π kilometers farther, you will end up at precisely the same place. Why? Because you will have gone full circle. And if you go 2π again, the same thing. Indeed if you go any multiple of 2π kilometers you will end up exactly where you started. And that means just as far north or south of Main Street, and just as far east or west of Main Avenue as you started. And what this means in terms of sine and cosine is that for any integer, n, it is always the case that

$$\sin(x) = \sin(x + 2n\pi) \tag{7.1b-3a}$$
$$\cos(x) = \cos(x + 2n\pi) \tag{7.1b-3b}$$

We are ready now to review a whole raft of useful relationships among trig functions. If you can't memorize them, you should learn to derive them quickly. They will soon become tools you will need to do homework and exam problems.

Here again are the two that we developed in the main text:

$$\sin^2(x) + \cos^2(x) = 1 \tag{7.1-2}$$

and

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{7.1-6}$$

From equation 7.1-2 you have immediately that

$$|\cos(x)| = \sqrt{1 - \sin^2(x)} \tag{7.1-1a}$$
$$|\sin(x)| = \sqrt{1 - \cos^2(x)} \tag{7.1-1b}$$

Now look at table 7-1. Notice that $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$. According to the odd and even properties of sine and cosine, we also know that $\sin\left(-\frac{\pi}{2}\right) = -1$ and $\cos\left(-\frac{\pi}{2}\right) = 0$. Suppose you let $b = \frac{\pi}{2}$ and stick it into equation 7.1-6.

$$\cos\left(a - \tfrac{\pi}{2}\right) = \cos(a)\cos\left(-\tfrac{\pi}{2}\right) - \sin(a)\sin\left(\tfrac{\pi}{2}\right) \tag{7.1b-4a}$$
$$\cos\left(a - \tfrac{\pi}{2}\right) = \cos(a) \times 0 - \sin(a) \times (-1) \tag{7.1b-4b}$$
$$\cos\left(a - \tfrac{\pi}{2}\right) = \sin(a) \tag{7.1b-4c}$$

And because cosine is an even function, it follows as well that

$$\cos\left(\tfrac{\pi}{2} - a\right) = \sin(a) \tag{7.1b-4d}$$

Both of these are true for any real number, a. Now substitute $u = \frac{\pi}{2} - a$ into equation 7.1b-4d and you have

$$\cos(u) = \sin\left(\tfrac{\pi}{2} - u\right) \tag{7.1b-4e}$$

Which is true for any real number, u. But suppose instead you substituted $u = a - \frac{\pi}{2}$ into equation 7.1b-4c. You'd get

$$\cos(u) = \sin\left(u + \tfrac{\pi}{2}\right) \tag{7.1b-4f}$$

which again is true for any real number, u.

Now lets take equation 7.1-6 and everywhere you see an a, replace it with $\frac{\pi}{2} - a$ and everywhere you see a b, replace it with $-b$

$$\cos\left(\tfrac{\pi}{2} - a - b\right) = \cos\left(\tfrac{\pi}{2} - a\right)\cos(-b) - \sin\left(\tfrac{\pi}{2} - a\right)\sin(-b) \tag{7.1b-5a}$$

Now simply apply the identities we have so far:

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \tag{7.1b-5b}$$

and we have a way of finding the sine of the sum of two angles. So now we have formulas for both sine and cosine for the sum of angles, but what about differences of angles? You can use the sum formulas together with the even and odd properties to, substituting $-b$ for b, to get:

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b) \tag{7.1b-6a}$$

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b) \tag{7.1b-6b}$$

You can also use the sum formulas to derive expressions for sine and cosine of double-angles. Simply observe that $2x = x + x$, and substitute into the sum expressions.

$$\cos(2x) = \cos(x + x) = \cos^2(x) - \sin^2(x) \tag{7.1b-7a}$$

$$\sin(2x) = \sin(x + x) = 2\sin(x)\cos(x) \tag{7.1b-7b}$$

You can do a sneaky trick on equation 7.1b-7a to get a half-angle by substituting $\frac{x}{2}$ and combining it with eq. 7.1-2.

$$\cos(x) + 1 = \cos\left(\tfrac{x}{2} + \tfrac{x}{2}\right) + 1 = \cos^2\left(\tfrac{x}{2}\right) - \sin^2\left(\tfrac{x}{2}\right) + 1 = \cos^2\left(\tfrac{x}{2}\right) - \sin^2\left(\tfrac{x}{2}\right) + \sin^2\left(\tfrac{x}{2}\right) + \cos^2\left(\tfrac{x}{2}\right) \tag{7.1b-8a}$$

When you cancel and simplify, you get

$$\cos(x) + 1 = 2\cos^2\left(\tfrac{x}{2}\right) \tag{7.1b-8b}$$

$$\sqrt{\frac{\cos(x) + 1}{2}} = \left|\cos\left(\tfrac{x}{2}\right)\right| \tag{7.1b-8c}$$

Likewise, you can change signs and get

$$1 - \cos(x) \ = \ 1 - \cos\left(\tfrac{x}{2} + \tfrac{x}{2}\right) \ = \ 1 - \cos^2\left(\tfrac{x}{2}\right) + \sin^2\left(\tfrac{x}{2}\right) \ = \ \sin^2\left(\tfrac{x}{2}\right) + \cos^2\left(\tfrac{x}{2}\right) - \cos^2\left(\tfrac{x}{2}\right) + \sin^2\left(\tfrac{x}{2}\right) \qquad (7.1b\text{-}9a)$$

When you cancel and simplify, you get

$$1 - \cos(x) \ = \ 2\sin^2\left(\tfrac{x}{2}\right) \qquad (7.1b\text{-}9b)$$

$$\sqrt{\frac{1-\cos(x)}{2}} \ = \ \left|\sin\left(\tfrac{x}{2}\right)\right| \qquad (7.1b\text{-}9c)$$

Putting in x instead of $\frac{x}{2}$, you can use the above to get formulas for sine squared and cosine squared:

$$\sin^2(x) \ = \ \frac{1-\cos(2x)}{2} \qquad (7.1b\text{-}9a)$$

$$\cos^2(x) \ = \ \frac{1+\cos(2x)}{2} \qquad (7.1b\text{-}9b)$$

(Observe what happens when you add the right-hand sides of the previous two equations)

Combining equations 7.1-6 and 7.1b-6a, you get

$$\cos(a+b) + \cos(a-b) \ = \ \cos(a)\cos(b) - \sin(a)\sin(b) + \cos(a)\cos(b) + \sin(a)\sin(b) \ = \ 2\cos(a)\cos(b) \qquad (7.1b\text{-}11a)$$

If you divide out the 2, you can see that taking the product of the cosines of two numbers is the same as taking half the cosine of their sum plus half the cosine of their difference. Likewise

$$\cos(a+b) - \cos(a-b) \ = \ \cos(a)\cos(b) - \sin(a)\sin(b) + -\cos(a)\cos(b) - \sin(a)\sin(b) \ = \ -2\sin(a)\sin(b) \qquad (7.1b\text{-}11b)$$

Again, if you divide out the -2, you can see that taking the product of the sines of two numbers is the same as taking half the cosine of their difference *minus* half the cosine of their sum. Similarly, you can combine equations 7.1b-5b and 7.1b-6b to get

$$\sin(a+b) + \sin(a-b) \ = \ \sin(a)\cos(b) + \cos(a)\sin(b) + \sin(a)\cos(b) - \cos(a)\sin(b) \ = \ 2\sin(a)\cos(b) \qquad (7.1b\text{-}12a)$$

$$\sin(a+b) - \sin(a-b) \ = \ \sin(a)\cos(b) + \cos(a)\sin(b) + -\sin(a)\cos(b) + \cos(a)\sin(b) \ = \ 2\cos(a)\sin(b) \qquad (7.1b\text{-}12b)$$

On any of the previous four identities you can substitute: $u = a + b$ and $v = a - b$, then you also have

$$a = \frac{u+v}{2} \quad \text{and} \quad b = \frac{u-v}{2}$$

If, for example, you put all that into eq. 7.1b-12b in place of a and b, you get the new identity:

$$\sin(u) - \sin(v) \; = \; 2 \, \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right) \tag{7.1b-12c}$$

You can also apply the same procedure to any of eq. 7.1b-11a and b and 7.1b-12a to get similar results. Try it.

Here's a cute one for you.

$$\begin{aligned} \sin(x) \; &= \sqrt{2}\left(\frac{\sqrt{2}\sin(x)}{2} + \frac{\sqrt{2}\cos(x)}{2}\right) \; = \\ &\sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right)\sin(x) + \sin\left(\tfrac{\pi}{4}\right)\cos(x)\right) \; = \\ &\sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right) \end{aligned} \tag{7.1b-13}$$

It's also equal to

$$\sqrt{2}\cos\left(\tfrac{\pi}{4} - x\right)$$

Tangent, Cotangent, Secant, and Cosecant

Surely you recall from trig that sine and cosine were not the only trig functions you studied. They also introduced you to

$$\tan(x) \; = \; \frac{\sin(x)}{\cos(x)} \tag{7.1b-14a}$$

$$\cot(x) \; = \; \frac{\cos(x)}{\sin(x)} \tag{7.1b-14b}$$

$$\sec(x) \; = \; \frac{1}{\cos(x)} \tag{7.1b-14c}$$

$$\csc(x) \; = \; \frac{1}{\sin(x)} \tag{7.1b-14d}$$

Figure 7-4 shows the geometric interpretation of these functions. Once again, the radius of the circle is 1. The angle, x, is still in radians. Observe that the point, **L**, has coordinates of $(\cos(x), \sin(x))$. We have

- length of $\overline{\mathbf{NK}}$ equal to $\tan(x)$.
- length of $\overline{\mathbf{MJ}}$ equal to $\cot(x)$.
- length of $\overline{\mathbf{OK}}$ equal to $\sec(x)$.
- length of $\overline{\mathbf{OJ}}$ equal to $\csc(x)$.

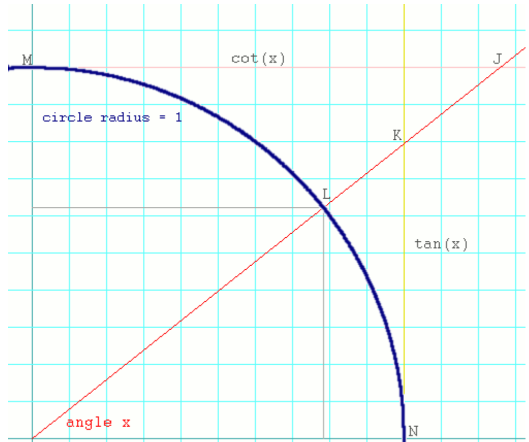


Fig. 7-4

Notice that the tangent function is so named because it is the length of a segment that is tangent to the circle. Likewise the cotangent.

Unlike the sine and cosine, these new functions are not continuous everywhere. For example, according to eq. 7.1b-14a, $\tan(x)$ has a $\cos(x)$ in its denominator. So everywhere that $\cos(x)$ is zero, $\tan(x)$ is discontinuous and undefined. This happens at odd multiples of $\frac{\pi}{2}$. Look at the blue trace in figure 7-5 to see what's going on here.

Similarly, equation 7.1b-14b shows that $\cot(x)$ has a $\sin(x)$ in its denominator, so everywhere $\sin(x)$ is zero, $\cot(x)$ is undefined and discontinuous. This happens at multiples of π. Look at the green trace in figure 7-5 to see what's going on in this case.

We shall be discussing more about continuity of trig functions in a later section.

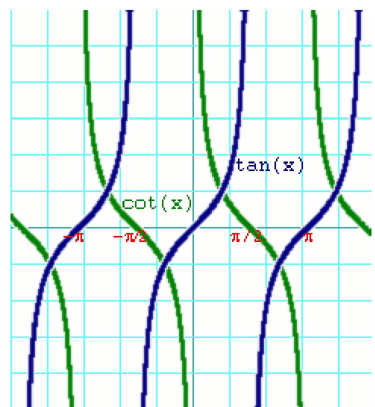
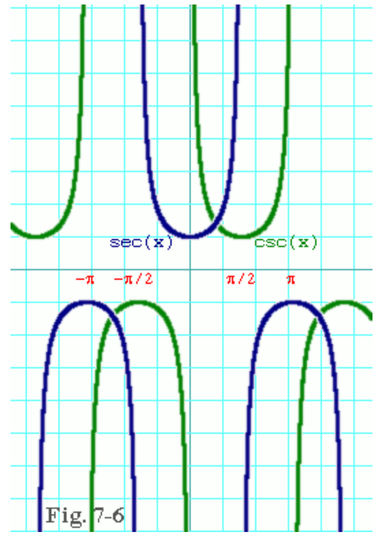


Fig 7-5



The functions, $\sec(x)$ and $\csc(x)$, follow a similar pattern. Observe in figure 7-6 that $\sec(x)$ (the blue trace) is discontinuous at odd multiples of $\frac{\pi}{2}$, and $\csc(x)$ (the green trace) is discontinuous at multiples of π. And the discontinuities occur for the very same reasons as they do in $\tan(x)$ and $\cot(x)$.

There are just a few identities that we'll go over concerning these functions. First, you recall from eq. 7.1b-14c that

$$\sec(x) = \frac{1}{\cos(x)}$$

so it must also be true that

$$\sec^2(x) = \frac{1}{\cos^2(x)}$$

But we all know from eq. 7.1-2 that the 1 in the numerator of the above is the same as $\sin^2(x) + \cos^2(x)$. So by replacing the numerator with that, we get

$$\sec^2(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} \qquad \text{(7.1b-15a)}$$

With just a little algebra and a glance back at eq. 7.1b-14a, you can see that

$$\sec^2(x) = \tan^2(x) + 1 \qquad \text{(7.1b-15b)}$$

and using the identical approach you can show that

$$\csc^2(x) = \cot^2(x) + 1 \qquad \text{(7.1b-15c)}$$

Developing a formula for $\tan(a+b)$ is a piece of cake when you use the formulas we already have for $\sin(a+b)$ and $\cos(a+b)$.

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} \qquad \text{(7.1b-16a)}$$

$$\tan(a+b) = \frac{\sin(a)\cos(b) + \sin(b)\cos(a)}{\cos(a)\cos(b) - \sin(a)\cos(b)} \qquad \text{(7.1b-16b)}$$

If you divide numerator and denominator of eq. 7.1b-16b by $\cos(a)\cos(b)$, you get some cancellations.

$$\tan(a+b) = \frac{\dfrac{\sin(a)}{\cos(a)} + \dfrac{\sin(b)}{\cos(b)}}{1 - \dfrac{\sin(a)\sin(b)}{\cos(a)\cos(b)}} \qquad \text{(7.1b-16c)}$$

Finally, applying eq. 7.1b-14a, you get the identity

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)} \qquad \text{(7.1b-16d)}$$

Using a very similar method you can come up one for the cotangent (which I will let you derive for yourself):

$$\cot(a+b) = \frac{\cot(a)\cot(b) - 1}{\cot(a) + \cot(b)} \tag{7.1b-16e}$$