7.2 As the World Turns

Derivatives of Trigonometric Functions

Let's get right back to the train that circles around New Atlantis. When the train is parked at Main Street and 10^{th} Avenue East, it faces due north. Suppose it rolled 0.1 km counterclockwise from that stop. How far north would it be. We already know from our discussion in the last section that it would be $\sin(0.1)$ km north of the intersection at Main Street and 10^{th} Avenue East. But look carefully at the diagram. As the train rolls out from the station, isn't it going *almost* due north. Indeed, during that entire 0.1 km of track isn't it going pretty close to north the whole way?

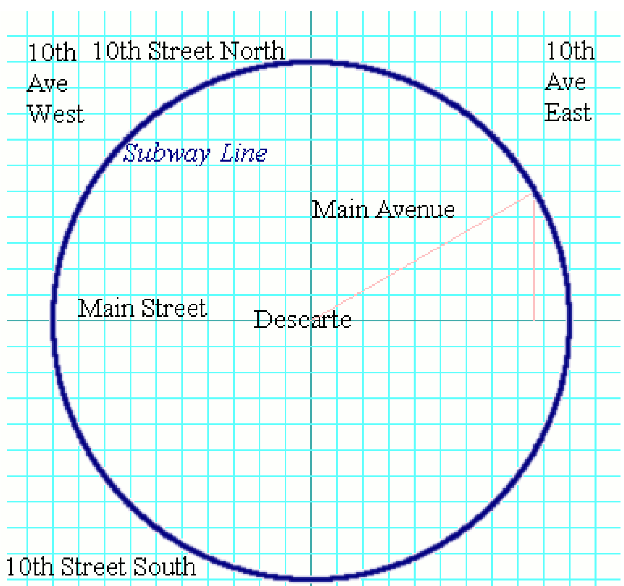


Fig. 7-1 New Atlantis Capital City

And so, for each meter of track the train covers in that first 0.1 km, doesn't it go almost a meter north? Another way of saying this would be to say that $\frac{\sin(0.1)}{0.1}$ is very nearly equal to one. My calculator gives 0.998334166 as the value of that expression carried out to 9 figures. That's pretty close to one.

Still, you have to agree that even in that first 0.1 km, the track does diverge slightly from due north. If you eyeball the diagram carefully you can even see it. But in the first 0.01 km, it diverges from due north even less. If you look at the diagram, in the first tenth of a block, you can't even discern any curvature of the track. It's extremely close to pointing due north the whole way. So if in the first 0.1 km the train was very nearly going one meter north for every meter around the track, in the first 0.01 km, it should be *even more nearly* going one meter north for every meter around the track. In other words, $\frac{\sin(0.01)}{0.01}$ ought to be closer to one than $\frac{\sin(0.1)}{0.1}$ was. And indeed, my calculator gives 0.999983333 for the value of $\frac{\sin(0.01)}{0.01}$. That's a whole lot closer to one than $\frac{\sin(0.1)}{0.1}$ was, and it was pretty close to begin with.

As you try shorter and shorter lengths of track from the station at Main Street and 10th Avenue East, our sense of geometry tells us that the the train will be going closer and closer to due north the entire way. If h is the length of track traveled from the station, then it seems that as h gets shorter and shorter, $\frac{\sin(h)}{h}$ should get closer and closer to one. To put this in mathematical terms, we have a sense that

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \tag{7.2-1}$$

But you should know by now that just having a sense of something is never enough in mathematics. To be sure of anything, you always need proof. And so we will now prove that equation 7.2-1 is true. Follow along carefully because this may be on an exam.

Between a Rock and a Hard Place

Look at figure 7-7. You can see a small right triangle inscribed in the circle whose interior is shaded yellow. The radius of the circle is one again. The angle at the center is h so the base of this triangle is $\cos(h)$ and the height is $\sin(h)$. According to the formula you already know for the area of a triangle $\left(A = \frac{1}{2}\text{height} \times \text{base}\right)$, the yellow area must be $\frac{1}{2}\sin(h)\cos(h)$.

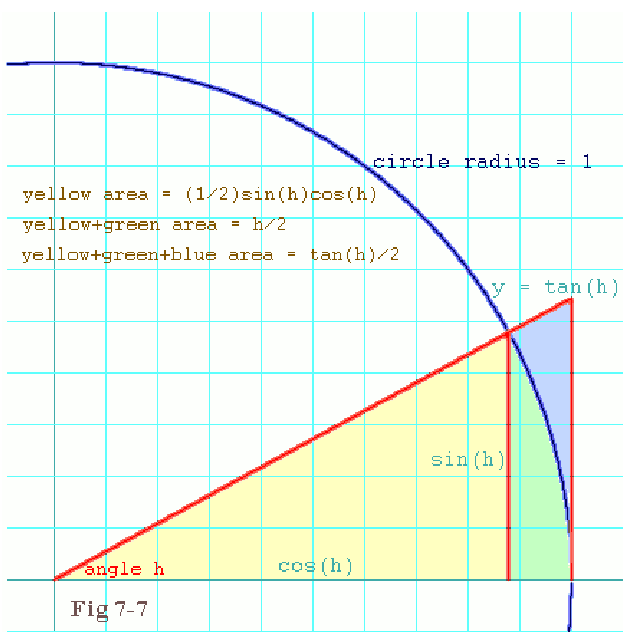


Fig 7-7

The circumscribed right triangle (that's the big one) that comprises the yellow, the green, and the blue shading, has a base equal to the radius of the circle, which is one, and a height of $\tan(h)$ (click here to see why this is). Once again, applying the formula for the area of a triangle you get that the combined yellow, green, and blue area is $\frac{1}{2}\tan(h)$.

But what about the area comprising just the yellow and the green shading? It is a pie-slice, and therefore represents the same fraction of the area of the entire circle that the angle, h, is of the entire circumference. In other words, because h is not only the angle in radians, but also the distance around the circumference subtended by that angle, it follows that h is to 2π radians as the area in question is to the area of the circle.

Once again, the radius of the circle is unity, so according the formula for the area of a circle ($A = \pi r^2$), the area of this circle is exactly π. So if a is the area of the pie-slice, then we have:

$$\frac{h}{2\pi} = \frac{a}{\pi} \tag{7.2-2}$$

The π's cancel, and hence the yellow-plus-green area must be exactly $\frac{h}{2}$.

Now observe that the yellow-plus-green-plus-blue area must be greater than the yellow-plus-green area, which in turn must be greater than the yellow area alone. In math symbols, that means

$$\frac{\tan(h)}{2} > \frac{h}{2} > \frac{\sin(h)\cos(h)}{2} \tag{7.2-3}$$

Now cancel the 2's and divide through by $\sin(h)$ (notice that $\sin(h)$ is positive, so we can divide the inequality by it. If you divided by a negative quantity, you would have to reverse the direction of the inequality signs). Since $\tan(h) = \frac{\sin(h)}{\cos(h)}$, you end up with

$$\frac{1}{\cos(h)} > \frac{h}{\sin(h)} > \cos(h) \tag{7.2-4}$$

Remember that when $\cos(h)$ is positive, it is always less than or equal to one, so $\frac{1}{\cos(h)}$ is always greater than or equal to one. And $\cos(h)$ is always positive when h is small. In fact, as h gets closer and closer to zero, $\cos(h)$ gets closer and closer to one. And, of course, so does $\frac{1}{\cos(h)}$, but from the other direction

When you take the limit as h goes to zero, you find that $\frac{h}{\sin(h)}$ is being *squeezed* between something just greater than one and something just less than one. And as h goes to zero, both the rock and the hard place close in on exactly one (this technique of trapping a limit between two things that approach each other is, in fact, called the squeeze theorem or the squeeze method). So $\frac{h}{\sin(h)}$ must be squeezed to that limit as well. And we conclude that

$$\lim_{h\to 0} \frac{1}{\sin(h)} \;=\; 1 \tag{7.2-5a}$$

It follows immediately that

$$\lim_{h\to 0} \frac{\sin(h)}{h} \;=\; 1 \tag{7.2-5b}$$

as well.

From this limit we can very easily prove another one that we will also need a little later. That is that

$$\lim_{h\to 0} \frac{1-\cos(h)}{h} \;=\; 0 \tag{7.2-5c}$$

This follows from the identity given in eq. 7.1b-9b (that is, $1-\cos(x) = 2\sin^2\left(\frac{x}{2}\right)$). This is because

$$\lim_{h\to 0} \frac{1-\cos(h)}{h} \;=\; \lim_{h\to 0} \frac{2\sin^2\left(\frac{h}{2}\right)}{h} \;=\; \lim_{h\to 0} \sin\left(\frac{h}{2}\right)\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)$$

As we've already seen, the right-hand factor (in the big parentheses) goes to 1 as $\frac{h}{2}$ goes to zero. But the other factor, just to its left, $\sin\left(\frac{h}{2}\right)$, goes to zero. And the limit of the product is the product of the limits.

So How Fast is the Train Going North?

Recall that the train travels around the track at 1 km per minute. Suppose nobody wants to get on or off the train at Main Street and 10^{th} Avenue East. So the train doesn't bother to stop there – it rolls right through the station at 1 km per minute. So what is its northward speed at that exact moment? Since the train is pointed due north at that moment, our intuition tells us that its northward speed must also be 1 km per minute as well. But now think of it in calculus terms. Remember that speed is the derivative of position with respect to time. If $x(t)$ is the trains east-west position as a function of time and $y(t)$ is its north-south position as a function of time, then the northward speed when the train rolls through the station will be $y'(t)$ at the moment, t, that it is at the station.

Let's call the moment that the train passes the station, $t = 0$. Then the train's northward speed at that moment is $y'(0)$. And according to our definition of the derivative,

$$y'(0) \;=\; \lim_{h\to 0} \frac{y(h) - y(0)}{h} \tag{7.2-6}$$

Now look at the diagram of the track once more. Can you see that as the train circles the track at 1 km per minute, its position at any time will be $(\cos(t), \sin(t))$? In other words, $x(t) = \cos(t)$ and $y(t) = \sin(t)$. And so

$$y'(t) = \frac{d \sin(t)}{dt} = \lim_{h \to 0} \frac{\sin(t+h) - \sin(t)}{h} \tag{7.2-7}$$

When you put in zero for t, equation 7.2-7 becomes easy because $\sin(0) = 0$. So it reduces to

$$y'(0) = \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \tag{7.2-8}$$

That confirms that the northward speed of the train as it rolls through the station is exactly 1 km per minute. But what about its northward speed at some other time besides $t = 0$? We have to take the limit shown in equation 7.2-7 at whatever t that happens to be. Fortunately we can apply a trig identity to $\sin(t+h)$. The resulting equation is

$$y(t) = \lim_{h \to 0} \frac{\sin(t)\cos(h) + \sin(h)\cos(t) - \sin(t)}{h} \tag{7.2-9a}$$

Breaking this up into three fractions, you get

$$y(t) = \lim_{h \to 0} \frac{\sin(t)\cos(h)}{h} + \frac{\sin(h)\cos(t)}{h} - \frac{\sin(t)}{h} \tag{7.2-9b}$$

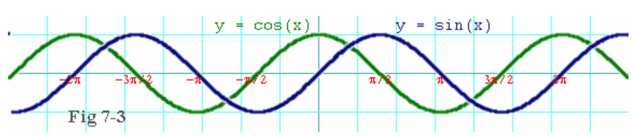
Now observe that if you group the first and third terms together and pull out the common factor, they are the same as $\sin(t)$ times the negative of the limit shown in eq. 7.2-5c, which, in the limit, is zero. So we can drop the first and third terms and be left with

$$y'(t) = \lim_{h \to 0} \frac{\sin(h)\cos(t)}{h} \tag{7.2-9c}$$

Furthermore, the $\frac{\sin(h)}{h}$ goes to one as h goes to zero (according to eq. 7.2-5b). And that leaves you with just

$$y'(t) = \cos(t) \tag{7.2-9d}$$

That's it. ***The derivative of the sine is the cosine.*** Look carefully at the graph of sine and cosine and try to see how the slope of the sine trace (in blue) is always exactly equal to the cosine trace (in green).



And what about the train's east-west speed at any time, t? Remember that the east-west position of the train is given by $x(t) = \cos(t)$. So

$$x'(t) = \lim_{h \to 0} \frac{\cos(t+h) - \cos(t)}{h} \tag{7.2-10a}$$

Again a trig identity comes to our aid. When you use the identity for $\cos(t+h)$, the above becomes

$$x'(t) = \lim_{h\to 0} \frac{\cos(t)\cos(h) - \sin(t)\sin(h) - \cos(t)}{h} \tag{7.2-10b}$$

Again we can group the first and third terms, extract the common factor, and apply eq. 7.2-5c to show that, in the limit as h goes to zero, the combination of those first and third terms goes to zero. So again you are left with only the middle term.

$$x'(t) = \lim_{h\to 0} \frac{-\sin(t)\sin(h)}{h} \tag{7.2-10c}$$

And again $\frac{\sin(h)}{h}$ goes to one as h goes to zero (according to eq. 7.2-5b), so

$$x'(t) = -\sin(t) \tag{7.2-10d}$$

And you're left with ***the derivative of the cosine is minus the sine.*** I urge you to study the graph in figure 7-3 again and see how the sine trace (in blue) is always the opposite of the slope of the cosine trace (in green).

Derivative of tan, cot, sec, and csc

In the trig identity section, you learned that these four functions were merely quotients of the sine and cosine functions. So to find the derivatives of tan, cot, sec, and csc, it would make sense to use the quotient rule. Let's begin by finding the derivative of $f(x) = \tan(x)$.

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)} \tag{7.2-11a}$$

Remembering that the derivative of the sine is the cosine and the derivative of the cosine is minus the sine, we apply the quotient rule to the quotient on the right to get:

$$f'(x) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \tag{7.2-11b}$$

or equivalently:

$$f'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \tag{7.2-11c}$$

You can simplify equation 7.2-11c in two different ways. One is to recognize the trig identity that you have in the numerator, namely that sine squared plus cosine squared of the same quantity is always equal to 1. That leaves you with the reciprocal of $\cos^2(x)$, which is the same as $\sec^2(x)$. The other simplification is to observe that the left-hand summand in the numerator is identical to the denominator. So if you break it up into the sum of two quotients, the left-hand quotient will be 1, and the right-hand quotient will be $\tan^2(x)$. Hence, here are the two simplifications:

$$f'(t) = \frac{1}{\cos^2(x)} = \sec^2(x) \tag{7.2-11d}$$

$$f'(t) = \frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x) \tag{7.2-11e}$$

The two forms (equation 7.2-11d and 7.2-11e) are identities of each other, and therefore entirely equivalent. So ***the derivative of the tangent is the secant squared.*** But it is also true ***that the derivative of the tangent is one plus the tangent squared.*** The latter is a very important property of the tangent function (that is to take its derivative you simply square it and add 1), and we will make use of it later on.

If you recall that $\cot(x)$ is simply the reciprocal of the tangent, you should be able to apply the same attack on it as we did above on the tangent function to show that ***the derivative of the cotangent is minus the cosecant squared,*** and that equivalently ***the derivative of the cotangent is minus one minus the cotangent squared.*** I urge you to take out pencil and paper right now and do this as an exercise.

Finding the derivative of secant and cosecant is even easier than doing so for tangent and cotangent. You can use the quotient rule if you like, or you can observe that the derivative of $f(x) = \frac{1}{x}$ is $f'(x) = -\frac{1}{x^2}$. Since secant and cosecant are reciprocals of cosine and sine respectively, you need only apply the chain rule and the above derivative to find their derivatives. For example, if $f(x) = \frac{1}{x}$ and $g(x) = \cos(x)$ and $h(x) = f(g(x)) = \frac{1}{\cos(x)}$, then the chain rule says

$$h'(x) = f'(g(x))\,g'(x) = \frac{-1}{\cos^2(x)}(-\sin(x)) = \tan(x)\sec(x) \tag{7.2-12}$$

So ***the derivative of the secant is the tangent times the secant.*** Once again, as an exercise, I urge you right now to use the same attack to show that ***the derivative of the cosecant is minus the cotangent times the cosecant.***

Derivative of Inverse Trig Functions

One way of finding the derivative of $y = \arcsin(x)$ would be to apply

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\arcsin(x+h) - \arcsin(x)}{h} \tag{7.2-14}$$

then use identities and tricks and algebra to work it through. Don't try it. They won't ask you to do that on any exam. The easy way to work this derivative is to use the chain rule. Simply observe that if equation $y = \arcsin(x)$ is true, then it must also be true that

$$\sin(y) = x \tag{7.2-15}$$

Now take the derivative of both sides, applying the chain rule to the left-hand side.

$$\cos(y)\, y' = 1 \tag{7.2-16}$$

But by the trig identity that you've memorized by now, you know that

$$\sin^2(x) + \cos^2(x) = 1 \tag{7.2-17}$$

or equivalently

$$\cos(y) = \sqrt{1 - \sin^2(x)} \tag{7.2-18}$$

And because equation 7.2-15 tells you that $\sin(y) = x$, you can substitute x anywhere you see $\sin(y)$.

$$\cos(y) = \sqrt{1 - x^2} \tag{7.2-19}$$

And you can substitute that expression for $\cos(y)$ back into equation 7.2-16 to get

$$\sqrt{1-x^2}\, y' = 1 \tag{7.2-20}$$

That makes it pretty easy to solve for y'.

$$y' = \frac{1}{\sqrt{1-x^2}} \tag{7.2-21}$$

And that is the derivative of $\arcsin(x)$. Go over it several times, since they might pop this derivation on you on the exam.

Interesting, isn't it, how a simple algebraic function made only from squares and square roots can be the derivative of something as abstract as an inverse trig function. What's happening here is that this derivative is telling us something about the relationship of $\sin(x)$ to its derivative. Think about it. Taking the derivative of $\sin(x)$ is equivalent to squaring it, subtracting the result from 1, then taking the square root of that. Of course, that doesn't work for any function. But for $\sin(x)$ it does. It's a special relationship that $\sin(x)$ enjoys with its own derivative. The concept of a function's derivative being related to the function itself according to some algebraic rule is an important one that you will be coming back to when you get to more advanced material. For now, just make a mental note of it and move on. And if the thought tickles your fancy, all the better.

Exercises

1) Use what you learned above about a) the derivative of the tangent function, and b) taking the derivative of an inverse function to determine the derivative of $f(x) = \arctan(x)$. **Hint:** In this instance, the more useful of the two forms for the derivative of the tangent function is the "one plus tangent squared" form. *Click here* to view solution.

2) Use the product rule to determine the derivative of $f(x) = \sin(x)\cos(x)$. Now observe the trig identity, $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$. Show that if you take the derivative of the right-hand side of this identity, it is consistent with the derivative you get using the left-hand side of the identity. *Click here* to view solution.

3) Use the product rule and the chain rule to find the derivative of

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

What can you say about $f'(x)$ when $x = 0$? *Click here* to view solution.

4) Use implicit differentiation on each of the following (where y is a function of x in each case):

a) $\sin(y) = \cos(x^2)$

b) $y\tan(x) = x$

c) $\sin(xy)\cos(x) = y^2$

d) $\tan(y) = \cos(2x)$

Click here to view solutions, but try to do them all before you do.

5) Find the limit:

$$\lim_{h\to 0} \frac{1-\cos(h)}{h}$$

Hint: Use the difference of squares and a common trig identity to convert this into limit having to do with $\sin(h)$. Then use what you know about the limit as h goes to zero of $\frac{\sin(h)}{h}$. When you are done, *click here*. Note that in the next section we will learn a much easier method for finding limits like this called L'Hopital's Rule.

6) Use figure 7-7 (shown here again for your convenience) to prove that for positive h in the first quadrant, it is always true that $\sin(h) < h$. Hint: Construct another line segment from the point $(\cos(h), \sin(h))$ to the point, $(1, 0)$, and see what the area of the resulting triangle is compared to the area of the pie-slice. Once you have proved that, use the inequality in a delta-epsilon proof that the function, $f(x) = \sin(x)$, is continuous at the point, $x = 0$. Take your best shot at this, and then *click here*.

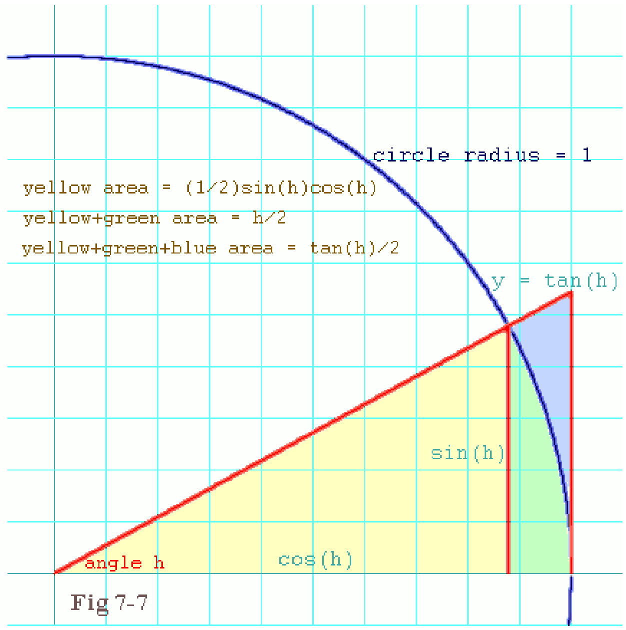


Fig 7-7

Proof that $\sin(x)$ is Continous Everywhere

In the last exercise above you proved that $f(x) = \sin(x)$ is continuous at $x = 0$. It would be better to know that $f(x) = \sin(x)$ is continuous for all x. And since there is an outside possibility that your instructor might spring something like this on an exam, here is a delta-epsilon proof of that, taken step by step. This is likely the hardest delta-epsilon proof that you will encounter in your first year of calculus, so if you go over this until you understand it, you will be ready for any delta-epsilon proof you are likely see on any exam.

The outline of the proof is that first we set up the delta-epsilon contract that we have to meet for

$$\lim_{x \to c} \sin(x) = \sin(c) \tag{7.2-22}$$

to be true for all x. Recall that this is the limit that *defines* continuity.

We will make a change of variables that puts the δ term to work invisibly inside the contract expression. You will see that once we set up the delta-epsilon contract in this way, there will be a trig identity that we can apply right away. We will then show that each of the terms in the resulting expression is less than some very simple expression. By combining the simple expressions, we will have an easy way to quantify the δ in terms of ϵ so that the contract must be met.

But first, some more basic inequalities concerning sine and cosine. You recall that for $-\frac{\pi}{2} < h < \frac{\pi}{2}$ (but h not equal zero) we showed (in problem 6) that it is always the case that $|\sin(h)| < |h|$. Observe also (using figure 7-3 if you need to) that it is always the case that $|\sin(h)| \leq 1$, and that $|\cos(h)| \leq 1$, no matter what h is. The following inequalities are immediate consequences of this (again assuming that h is not equal zero):

$$\cos^2(h) \ \leq \ |\cos(h)| \tag{7.2-23a}$$

$$1 - |\cos(h)| \ \leq \ 1 - \cos^2(h) \ = \ \sin^2(h) \ < \ h^2 \tag{7.2-23b}$$

or more succinctly,

$$1 - |\cos(h)| \ < \ h^2 \tag{7.2-23c}$$

Also please recall from algebra that for any a and b, it is always true that $|a \pm b| \leq |a| + |b|$. We will be using all these inequalities very shortly.

One last note. This may seem obvious, but it bears repeating before getting into the meat of this proof. And that is that the relation, $<$, is *transitive*. So is the relation, $\leq$. That means, for example, that if $a < b$ *and* $b < c$, *then* $a < c$. Likewise, if $a \leq b$ *and* $b \leq c$, *then* $a \leq c$. We will be using this over and over in the proof that follows to show that the contract is still met even when we replace the original expressions with new ones that are greater.

Step 1: Write the contract. To prove the limit shown in equation 7.2-22, the contract is that whatever $\epsilon > 0$ I might name, you need to be able to name a $\delta > 0$ so that

$$|\sin(x) - \sin(c)| \ < \ \epsilon \tag{7.2-24}$$

whenever $|x - c| \ < \ \delta$. In other words, no matter how close I demand that $\sin(x)$ be to $\sin(c)$, you can find an x close enough to c to make it happen. The degree of closeness of x to c is expressed by the δ. How close I am demanding $\sin(x)$ to be to $\sin(c)$ is expressed by the ϵ.

Step 2: A change of variable. Observe that if $|x - c| \leq \delta$, then there is some h, where $0 < h \leq \delta$, that makes is so that $x = c \pm h$. So we can substitute $c \pm h$ into eq. 7.2-24 wherever we see x.

$$|\sin(c \pm h) - \sin(c)| \ < \ \epsilon \tag{7.2-25}$$

Step 3: Apply the identity for the sine of a sum (or difference). You remember, $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ and $\sin(a - b) = \sin(a)\cos(b) - \sin(b)\cos(a)$. It doesn't take much of a stretch to see that you can combine these two rules into

$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$. So we apply that last form to eq. 7.2-25:

$$|\sin(c)\cos(h) \pm \sin(h)\cos(c) - \sin(c)| < \epsilon \tag{7.2-26a}$$

You can combine the first and third terms that are inside the absolute value sign to get

$$|\sin(c)(\cos(h) - 1) \pm \sin(h)\cos(c)| < \epsilon \tag{7.2-26b}$$

Step 4: Use the absolute value inequality. If $|a \pm b| \leq |a| + |b|$, and we can choose a and b in such a way that we guarantee $|a| + |b| < \epsilon$, then that would force $|a \pm b| \leq |a| + |b| < \epsilon$. In this case, $a = \sin(c)(\cos(h) - 1)$ and $b = \sin(h)\cos(c)$. Hence if we can choose h so that

$$|\sin(c)(\cos(h) - 1)| + |\sin(h)\cos(c)| < \epsilon \tag{7.2-26c}$$

that would guarantee our contract. Notice that

$$|\sin(c)(\cos(h) - 1)| = |\sin(c)(1 - \cos(h))|$$

So

$$|\sin(c)(1 - \cos(h))| + |\sin(h)\cos(c)| < \epsilon \tag{7.2-26d}$$

is the same thing.

Step 5: Restrict the range of h. This is the same as restricting the range of δ. But the contract allows us to make δ as close to zero as we like. So there is no problem restricting it to $0 < \delta < 1$. Which, of course, restricts h to a range of $0 < h < 1$. Why would we want to do that? So that we keep things like $\cos(h)$ within a range where its behavior is simple. In this case we are interested in keeping $\cos(h)$ positive. You will see why momentarily.

Step 6: Substitute simpler terms that are known to be greater. Once again, if something greater than our contract expression is less than ϵ, then that forces the contract expression to be less than ϵ as well (because of *transitivity*). So, for example, we know that if $0 < \cos(h) \leq 1$, then $\cos(h) = |\cos(h)|$. And we know from the discussion above that $1 - |\cos(h)| \leq h^2$. Consequently wherever we see $1 - \cos(h)$, we can replace it with h^2 and rest assured that replacement is at least as large as what it replaced. Hence our contract remains satisfied if

$$\left|\sin(c)\, h^2\right| + |\sin(h)\cos(c)| < \epsilon \tag{7.2-28a}$$

Likewise, since $|\sin(h)| < h$ and h is known to be positive, we can replace $\sin(h)$ with h, knowing that it won't make the expression any smaller. Hence

$$\left|\sin(c)\, h^2\right| + |h\cos(c)| \; < \; \epsilon \tag{7.2-28b}$$

still guarantees the contract. We also know that no matter what c is, $|\sin(c)| \leq 1$ and $|\cos(c)| \leq 1$. So wherever we see them, we can replace them with 1 and still guarantee the contract:

$$\left|h^2\right| + |h| \; < \; \epsilon \tag{7.2-28c}$$

And since h is positive, this becomes simply

$$h^2 + h \; < \; \epsilon \tag{7.2-28d}$$

At this point, we could just go and solve the quadratic inequality above. But there is an easier way. Remember that $h < 1$. Consequently $h^2 < h$ (remember that this works because we know h to be positive). So by the same thinking as was responsible for all our other replacements above, we have

$$h + h \; = \; 2h \; < \; \epsilon \tag{7.2-28e}$$

or equivalently

$$h \; < \; \frac{\epsilon}{2} \tag{7.2-28f}$$

If you can choose h so that it satisfies this inequality, this still guarantees the contract.

Step 7: Bring back the δ. Since $h \leq \delta$, you can replace h with δ, and the inequality still guarantees the contract. So in the end, whatever ϵ I choose, you need only give me a δ that is less than half of ϵ *and* no more than 1 (remember we restricted the range of h before), and the contract is met.

And that is the proof that $\sin(x)$ is continuous. Admittedly it's a lot to swallow all at once. But go back and reread the outline of the proof up at the beginning. Then see how each step works into something in the outline. The outline tells you through which towns the proof passes. The detailed steps tell you where to make your right and left turns so that you end up unerringly at your destination. If you think about complicated proofs that way, you will never get lost.