11.7 Partial Fractions

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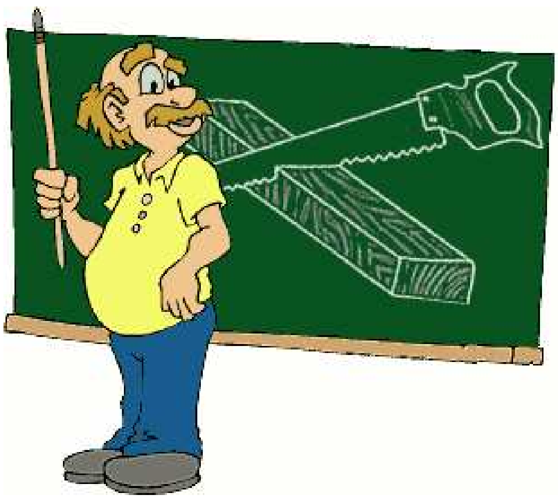
(divide and conquer)



In high school algebra you learned to add quotients of expressions by putting them over a common denominator. For example

$$\frac{1}{x-1} + \frac{1}{x+5} = \frac{x+5+x-1}{(x-1)\,(x+5)} = \frac{2x+4}{x^2+4x-5} \qquad (11.7\text{-}1)$$

Now that we are doing integrals, it is useful to make complicated expressions, like the one to the right of the equal in 11.7-1, into the sum of simpler expressions, like what's on the left of the equal in 11.7-1. The integral of what's to the left of the equal is

$$\int \left(\frac{1}{x-1} + \frac{1}{x+5} \right) dx = \int \frac{dx}{x-1} + \int \frac{dx}{x+5}$$
$$= \ln|x-1| + \ln|x+5| + C \qquad (11.7\text{-}1a)$$

You can see that the left-hand version of 11.7-1 is much easier to integrate than the right-hand version. But since the two sides of 11.7-1 are equal, their integrals must be equal as well. So if you were presented with having to integrate the expression on the right-hand side of equation 11.7-1, it would be useful to know the algebraic trick that turns it into the sum shown on the left-hand side of 11.7-1. That is what the method of *partial fractions* is all about. Through it we do the whole common denominator thing backward.

Using the method of partial fractions it is possible to integrate any rational function (that is any function that is the quotient of two polynomials) provided that you can factor the denominator (and remember that *The Fundamental Theorem of Algebra* guarantees that every polynomial can be factored entirely into first and second degree polynomials). Of course factoring high-degree polynomials is anything but trivial. But I'm sure your instructor understands that and will only assign you problems where the denominator is easily factored.

In this section I will show you two different methods for converting a rational function into partial fractions. One is the method taught in most beginning calculus classes. The other was invented by a fellow named Heaviside (the same person who predicted that the earth's atmosphere would reflect radio waves and after whom the reflective layer of the atmosphere is named). To see a brief biography of Oliver Heaviside, click here.The second method does not work on as wide a range of problems as the first, but on the ones that it does work, it requires much less computation and is less prone to human error.

Let's do an example. Let's use the same denominator as the right-hand side of equation 11.7-1, but a different numerator.

$$\int \frac{8x+22}{x^2+4x-5}dx = \int \frac{8x+22}{(x-1)(x+5)}dx \qquad (11.7\text{-}2)$$

We are looking for a way to break this integrand up into the sum of something over $(x-1)$ plus something over $(x+5)$. The problem is that we don't know what those two somethings are. So what we do is what you learned to do in algebra. That is, treat the two somethings as unknowns and solve for them. We'll call them, A and B. The equation they must solve is

$$\frac{8x+22}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5} \qquad (11.7\text{-}3)$$

The method that is commonly taught goes like this. We already know that the common denominator is $(x-1)(x+5)$. To put A over that common denominator, we would have to multiply it by $(x+5)$. To put B over that same common denominator we would have to multiply it by $(x-1)$. And when you add those two products up, it better come out equal to the numerator on the left, that is, $8x+22$. Why? Because the numerator on the left is already over the common denominator of $(x-1)(x+5)$. Equating just the numerators gives us the equation

$$A(x+5) + B(x-1) = 8x+22 \qquad (11.7\text{-}4a)$$

or equivalently

$$Ax + 5A + Bx - B = 8x + 22 \qquad (11.7\text{-}4b)$$

Clearly all the terms on the left that are multiplied by x must add up to $8x$, and all the constant terms must add up to 22. Make sure you see why this is. So from equation 11.7-4b we can extract two equations:

$$Ax + Bx = 8x \tag{11.7-5a}$$

and

$$5A - B = 22 \tag{11.7-5b}$$

From the first of these you can divide out the x to simplify it to

$$A + B = 8 \tag{11.7-5c}$$

Now we have two linear equations in two unknowns. You should remember how to solve these from high school algebra. When you solve them simultaneously you find that $A = 5$, and $B = 3$. Hence

$$\begin{aligned}\int \frac{8x+22}{x^2+4x-5}dx &= \int \frac{5\,dx}{x-1} + \frac{3\,dx}{x+5} \\ &= 5\ln|x-1| + 3\ln|x+5| + C\end{aligned} \tag{11.7-6}$$

This one isn't so bad to do the traditional way because you have only two equations in two unknowns. But if you have higher degree polynomials, you will have more equations in more unknowns to solve, hence it will take more time and you will be more likely to make a mistake. Heaviside recognized this and came up with an alternative method for solving for A and B. His method has the simplicity of generating A and B one at a time. Here is equation 11.7-3 again

$$\frac{8x+22}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$$

Heaviside tells us that to find A, multiply both sides by $(x-1)$ (that is the binomial under the A).

$$\left.\frac{8x+22}{(x+5)} = A + \frac{B(x-1)}{x+5}\right]_{x=1} \tag{11.7-7a}$$

The bracket with the $x = 1$ means to evaluate the whole thing at $x = 1$. Notice that that value of x is precisely what will make the B expression become zero.

$$\frac{8+22}{1+5} = A + \frac{0\,B}{1+5} = A = 5 \tag{11.7-7b}$$

As easy as that, we get a solution for A without ever having to do simultaneous equations. To find B we multiply equation 11.7-3 by $(x+5)$ (the binomial under B) and evaluate it at $x = -5$, which is precisely the value of x that will drop A out of the equation.

$$\left.\frac{8x+22}{(x-1)} = \frac{A(x+5)}{x-1} + B\right]_{x=-5} \tag{11.7-7c}$$

and lo

$$\frac{-40+22}{-6} = \frac{0\,A}{x-1} + B = B = 3 \tag{11.7-7d}$$

out pops the solution for B.

Let's do a slightly harder example

$$\int \frac{x^2+4x-33}{(x+1)(x-2)(x-3)}dx \tag{11.7-8a}$$

Step 1: Write out the partial fractions with unknown numerators. This is the first step no matter which of the two methods we use.

$$\int \frac{x^2+4x-33}{(x+1)(x-2)(x-3)}dx = \int \left(\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3}\right) dx \tag{11.7-8b}$$

Standard method, step 2: Put all the partial fractions over the common denominator. This means multiplying A by $(x-2)(x-3) = x^2-5x+6$, multiplying B by $(x+1)(x-3) = x^2-2x-3$, and multiplying C by $(x+1)(x-2) = x^2-x-2$. When we sort it all out and gather terms of like powers of x, we get

$$\int \frac{x^2+4x-33}{(x+1)(x-2)(x-3)}dx = \tag{11.7-9}$$
$$\int \frac{(A+B+C)\,x^2 + (-5A-2B-C)\,x + (6A-3B+2C)}{(x+1)(x-2)(x-3)}dx$$

Standard method, step 3: Equate the numerators' coefficients according to powers of x. That is, in the left-hand numerator, the coefficient for the x^2 term must be equal to whatever the coefficient of the x^2 term is in the right-hand numerator. Same thing for the coefficients of the x terms and again for the constant terms.

(11.7-10)

$$\begin{array}{rcrcrcrl} 1 & = & A & + & B & + & C & \text{equating coefficients of the } x^2 \text{ terms} \\ 4 & =- & 4A & - & 2B & - & C & \text{equating coefficients of the } x \text{ terms} \\ -33 & = & 6A & - & 3B & - & 2C & \text{equating coefficients of the constant terms} \end{array}$$

This gives us linear equations for the unknowns, A, B, and C.

Standard method, step 4: Solve the linear equations. You should know how to do this and come up with numbers for A, B, and C by solving the three linear equations simultaneously. If you don't, you can see how I did it by clicking here. The result we get is $A = -3$, $B = 7$, and $C = -3$.

Either method, last step: Put in the solved values of the coefficients and integrate.

$$\int \left(-\frac{3}{x+1} + \frac{7}{x-2} - \frac{3}{x-3} \right) dx = \qquad (11.7\text{-}11)$$
$$-\,3\ln|x+1| \;+\; 7\ln|x-2| \;-\; 3\ln|x-3| \;+\; K$$

Notice that I used K instead of C here for the undetermined constant. Not a big deal except for sticklers on matters of form. The reason is that I already used the C symbol as one of the unknown coefficients. So it removes confusion to supply a different name for the undetermined constant.

Now we will do the same example using Heaviside's method. Here again is equation 11.7-8b:

$$\int \frac{x^2+4x-33}{(x+1)\,(x-2)\,(x-3)} dx = \int \left(\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3} \right) dx$$

Heaviside method, step 2: Solve for A by multiplying equation 11.7-8b through by $(x+1)$ and evaluating at $x=-1$.

$$\frac{x^2+4x-33}{(x-2)\,(x-3)} = A + \frac{(x+1)\,B}{x-2} + \frac{(x+1)\,C}{x-3}\Bigg]_{x=-1} = \frac{1-4-33}{-3\times -4}$$

Because we multiplied by $(x+1)$ while $x=-1$, we eliminated the B and C terms (that is we multiplied them by zero). So we find from this equation alone that $A=-3$.

Heaviside method, step 3: Solve for B by multiplying equation 11.7-8b through by $(x-2)$ and evaluating at $x=2$.

$$\frac{x^2+4x-33}{(x+1)\,(x-3)} = \frac{A\,(x-2)}{x+1} + B + \frac{C\,(x-2)}{x-3}\Bigg]_{x=2} = \frac{4+8-33}{3\times -1}$$

This eliminates the A and C terms by multiplying them by zero. We're left with $B=7$ from this equation alone.

Heaviside method, step 4: Solve for C by multiplying equation 11.7-8b through by $(x-3)$ and evaluating at $x=3$.

$$\frac{x^2+4x-33}{(x+1)\,(x-2)} = \frac{A\,(x-3)}{x+1} + \frac{B\,(x-3)}{x-2} + C\Bigg]_{x=3} = \frac{9+12-33}{4\times 1}$$

This eliminates the A and B terms by multiplying them by zero. We're left with $C=-3$ from this equation alone. Now simply go the the last step (11.7-11), which is the same for both methods.

Dealing with Improper Fractions

Either of the methods we have discussed for breaking a rational function into partial fractions *requires that the degree of the numerator be less than the degree of the denominator*. Rational functions that do not meet this criterion are called *improper fractions*. If you try to apply either of the methods to any improper fraction, you will run into trouble before you are done. This is not a problem, though, since a little algebra can whip any improper fraction into a proper one. All you need to do is some polynomial long division. For example, if you have

$$\int \frac{2x^3 - 7x^2 + 6x - 21}{(x+1)(x-2)(x-3)}\,dx \tag{11.7-12a}$$

The numerator is a cubic (polynomial of degree 3). When you multiply out the denominator you get $(x+1)(x-2)(x-3) = x^3 - 4x^2 + x + 6$, which is also a polynomial of degree 3. This fails the test of the numerator being of lesser degree than the denominator. So we employ polynomial long division.

$$\begin{array}{r|r} & 2 \\ \hline x^3 - 4x^2 + x + 6 & 2x^3 - 7x^2 + 6x - 21 \\ & -\,2x^3 + 8x^2 - 2x - 12 \\ \hline & x^2 + 4x - 33 \end{array}$$

According to the rule about quotients, denominators, and remainders, the integral becomes

$$\int \frac{2x^3 - 7x^2 + 6x - 21}{(x+1)(x-2)(x-3)}dx = \tag{11.7-12b}$$

$$\int \left(2 + \frac{x^2 + 4x - 33}{x^3 - 4x^2 + x + 6}\right) dx = 2\int dx + \int \frac{x^2 + 4x - 33}{(x+1)(x-2)(x-3)}\,dx$$

That leaves us with the sum of an easy integral with one that meets the criterion for partial fractions (and is the same as the one we did in the last paragraph).

Dealing with Repeated Roots

In each of the partial fraction problems we have done so far, the polynomial in the denominator has had distinct real roots (that is real roots that are all different from each other). Here we will deal with the complication that confronts us if two or more of the real roots are the same. You will see that whether you do it using the standard method or the Heaviside method, there is a new wrinkle to setting it up. Once we have it set up, the standard method goes the same way as before. The Heaviside method, though, becomes a little more complicated.

Here is an example

$$\int \frac{x^2 - 2x + 17}{(x+3)(x-1)^2}\,dx \tag{11.7-13a}$$

You can see that the root in the denominator at $x = 1$ is occurs twice.

Step 1: The way you set this one up for partial fractions is

$$\frac{x^2 - 2x + 17}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \tag{11.7-13b}$$

Note *that the number of unknowns on the right must be equal to the degree of the denominator polynomial on the left.* In this case that denominator polynomial is a cubic. So we have to have three unknowns. But there are only two roots (one of them repeated). In order to set it up for three unknowns, we use various powers of the repeated root in the denominators of the partial fractions. The highest power of each root used is equal to the number of times it occurs in the denominator of the polynomial on the left. In this case $(x-1)$ occurs twice in the denominator of the left-hand side. So there must be a partial fraction with that root to the first power and another with that root to the second power, just as we have it in equation 11.7-13b. Since the $(x+3)$ root appears only once on the left, it results in only one partial fraction, with that root taken to the first power, on the right.

Standard method, step 2: Put all the partial fractions over a common denominator. So we multiply A by $(x-1)^2 = x^2 - 2x + 1$. We multiply B by $(x+3)(x-1) = x^2 + 2x - 3$. And we multiply C by $(x+3)$. Gathering the like powers of x, as we did in the last example, we get

$$\frac{x^2 - 2x + 17}{(x+3)(x-1)^2} = \frac{(A+B)x^2 + (-2A + 2B + C)x + (A - 3B + 3C)}{(x+3)(x-1)^2} \tag{11.7-14}$$

Standard method, step 3: Equate the numerators' coefficients according to powers of x.

(11.7-15)

$1 =$	$A + B$	equating coefficients of the x^2 terms
$-2 =$	$-2A + 2B + C$	equating coefficients of the x terms
$17 =$	$A - 3B - 3C$	equating coefficients of the constant terms

This gives us linear equations for the unknowns, A, B, and C.

Standard method, step 4: Solve the linear equations. We get $A = 2$, $B = -1$, and $C = 4$. If you need to see how to find this solution, click here.

Either method, last step: Put in the solved values of the coefficients and integrate.

$$\int \left(\frac{2}{x+3} - \frac{1}{x-1} + \frac{4}{(x-1)^2} \right) dx \qquad (11.7\text{-}16)$$
$$2 \ln |x+3| - \ln |x-1| - \frac{4}{x-1} + K$$

You can solve this same example using the Heaviside method, but because of the repeated root, it is not as easy as ABC the way the last one was. Here is equation 11.7-13b again:

$$\frac{x^2 - 2x + 17}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Heaviside method, step 2: Go for the highest power first. That means we solve for C first. To do that, multiply equation 11.7-13b by $(x-1)^2$, and evaluate it at $x = 1$.

$$\frac{x^2 - 2x + 17}{(x+3)} = \frac{A(x-1)^2}{x+3} + B(x-1) + C \Bigg]_{x=1} = \qquad (11.7\text{-}17)$$
$$\frac{0A}{x+3} + 0B + C = \frac{1 - 2 + 17}{4} = 4$$

Can you see how multiplying by the highest power denominator and evaluating at its root eliminates everything except the one partial fraction we are trying to solve? That lets us establish with this equation alone that $C = 4$.

Heaviside method, step 3a: Subtract out the partial fraction you just solved. This is the tricky part of Heaviside when there are repeated roots. We know that $C = 4$. So equation 11.7-13b becomes

$$\frac{x^2 - 2x + 17}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{4}{(x-1)^2} \qquad (11.7\text{-}18a)$$

Now put the partial fraction we just solved for over the common denominator and subtract it out of both sides. That drops it completely from the right and subtracts $4(x+3)$ from the numerator on the left.

$$\frac{x^2 - 6x + 5}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} \qquad (11.7\text{-}18b)$$

Heaviside method, step 3b: Simplify the left side using polynomial long division. Observe that the left side's denominator has a $(x-1)^2$ in it. The right hand side has only a $(x-1)$ to the first power in it. So it must be the case that on the left, the numerator and denominator have a common factor of $(x-1)$. We use polynomial long division to factor it out.

$$\begin{array}{r}
x-5 \\
x-1 \overline{\big) \; x^2-6x+5} \\
\underline{-x^2 \;\; +x \quad\;\;} \\
-5x+5 \\
\underline{5x-5} \\
0
\end{array}$$

Notice that it divides out perfectly with no remainder. *It should always do that* in this step of the Heaviside method for repeated roots. If it doesn't (that is, if you do get a nonzero remainder), then you made a mistake somewhere, and you should go back and check your work. With factoring out the common $(x-1)$ from numerator and denominator of equation 11.7-18b, it becomes

$$\frac{x2-6x+5}{(x+3)\,(x-1)^2} = \frac{x-5}{(x+3)\,(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} \qquad (11.7\text{-}18c)$$

The result is that you have now reduced the problem by one power of the repeated root. Since the repeated root occurred twice in the original problem, it now occurs only once in the reduced problem. That means you can do the rest of the problem using the Heaviside method for distinct roots, which is easier than what we just did.

Heaviside method, step 4: Solve for A by multiplying equation 11.7-18c through by $(x+3)$ and evaluating at $x=-3$.

$$\left.\frac{x-5}{x-1} = A + \frac{B\,(x+3)}{x-1}\right]_{x=-3} = \qquad (11.7\text{-}19a)$$

$$A + \frac{0B}{x-1} = \frac{-3-5}{-3-1} = 2$$

This eliminates B by multiplying it by zero. We're left with $A=2$ from this equation alone.

Heaviside method, step 5: Solve for B by multiplying equation 11.7-18c through by $(x-1)$ and evaluating at $x=1$.

$$\left.\frac{x-5}{x+3} = \frac{A\,(x-1)}{x+3} + B\right]_{x=1} = \qquad (11.7\text{-}19b)$$

$$\frac{0A}{x-1} + B = \frac{1-5}{1+3} = -1$$

This eliminates A by multiplying it by zero. We're left with $B=-1$ from this equation alone. We now have solutions for all three partial fraction coefficients (that is, $A=2$, $B=-1$, and $C=4$). The last step is the same for both methods.

Dealing with Nonreal Roots

So far we have dealt with rational functions whose denominators factor completely into binomials. But as you know, not every polynomial is so compliant. With some of them we find that one or more of the factors are quadratics that have no real roots. How do you apply the method of partial fractions to these? I'll show you by an example of one that has just one such factor.

$$\int \frac{x^2 - 3x + 46}{(x+3)(x^2 - 2x + 17)}\, dx \tag{11.7-20a}$$

If you try to factor the $x^2 - 2x + 17$ further by applying the quadratic formula, you find that you get a negative number under the radical. So we are stuck with this one the way it is. How do you break it into partial fractions?

Step 1: Set it up.

$$\frac{x^2 - 3x + 46}{(x+3)(x^2 - 2x + 17)} = \frac{A}{x+3} + \frac{Bx + C}{x^2 - 2x + 17} \tag{11.7-20b}$$

It is still true that the number of unknowns on the right must equal the degree of the denominator polynomial on the left. That is a cubic, so we need three unknowns. When a partial fraction has an irreducible quadratic in its denominator ("irreducible" is just math-speak for "cannot be factored"), it always gets two unknowns in its numerator, according to the scheme you see in equation 11.7-20b.

Standard method, step 2: Put all the partial fractions over a common denominator. Hence the A gets multiplied by $(x^2 - 2x + 17)$. The $Bx + C$ gets multiplied by $(x + 3)$, yielding $Bx^2 + (3B + C)\, x + 3C$. When we gather like powers of x we get

$$\frac{x^2 - 3x + 46}{(x+3)(x^2 - 2x + 17)} = \frac{(A+B)\, x^2 + (-2A + 3B + C)\, x + (17A + 3C)}{(x+3)(x^2 - 2x + 17)} \tag{11.7-21}$$

Standard method, step 3: Equate the numerators' coefficients according to powers of x.

(11.7-22)

$1 = A + B$		equating coefficients of the x^2 terms
$-3 = -2A + 3B + C$		equating coefficients of the x terms
$46 = 17A + 3C$		equating coefficients of the constant terms

This gives us linear equations for the unknowns, A, B, and C.

Standard method, step 4: Solve the linear equations. We get $A = 2$, $B = -1$, and $C = 4$ (same solution as before – hmmm. I wonder if Karl's cooking the numbers). If you need to see how to find this solution, click here.

Either method, last step: Put in the solved values of the coefficients and integrate.

$$\int \left(\frac{2}{x+3} + \frac{-x+4}{x^2-2x+17} \right) dx = \qquad (11.7\text{-}23)$$
$$2\int \frac{dx}{x+3} - \int \frac{x\,dx}{x^2-2x+17} + 4\int \frac{dx}{x^2-2x+17}$$

Each of the three integrals on the second line of the above equation is similar to ones we did in previous sections. The first is trivial. The second and third require that you complete the square.

$$v = x-1 \quad \text{and} \quad v^2 = x^2-2x+1 \quad \text{and} \quad v^2+16 = x^2-2x+17$$

Again $dv = dx$.

$$\int \left(\frac{2}{x+3} + \frac{-x+4}{x^2-2x+17} \right) dx = \qquad (11.7\text{-}23a)$$
$$2\int \frac{dx}{x+3} - \int \frac{(v+1)\,dv}{v^2+16} + 4\int \frac{dv}{v^2+16} =$$
$$2\int \frac{dx}{x+3} - \int \frac{v\,dv}{v^2+16} + 3\int \frac{dv}{v^2+16} =$$

The second integral is then susceptible to simple substitution. And the third requires trig substitution. The substitution for the second integral is.

$$s = v^2+16 \quad \text{and} \quad \tfrac{1}{2}ds = v\,dv$$

The substitution for the third integral is

$$\tan(u) = \frac{v}{4} \quad \text{and} \quad 4\left(\tan^2(u)+1\right) du = dv$$

I won't take you through all the steps individually because that is material covered in previous sections. When we do them all we end up with

$$2\ln|x+3| - \tfrac{1}{2}\ln\left|x^2-2x+17\right| + \tfrac{3}{4}\arctan\left(\frac{x-1}{4}\right) + K \qquad (11.7\text{-}24)$$

So what about applying Heaviside's method to this one? It turns out that when you have nonreal roots, Heaviside's method can only identify the coefficients that go with the real roots. In the problem we just did using the traditional method, there was one real root, and Heaviside's method can find it.

The truth is that Heaviside's method can indeed find coefficients that go with the nonreal roots too, but you have to use complex-number arithmetic. The nonreal roots turn out to be complex numbers. You can use them to factor a quadratic that was irreducible under the real numbers. Then you can use complex-number arithmetic to do Heaviside's method on those complex roots the same way we did with the real roots. After that you still have to recombine each pair of complementary partial fractions that you get to recreate a quadratic partial fraction with real coefficients. It's a lot of detail work. **I don't recommend that you do any of your assigned classroom problems this way.** You might confuse your instructor. Also doing complex arithmetic by hand is prone to mistakes. But if you insist on trying this method on your own to satisfy your curiosity, you can find complex number calculators (which are helpful) on the web. Click here to see how to download them to your PC (sorry, I don't know of any that run on a Mac or under X-Windows) You can also try the Online Complex Calculator.

Here is equation 11.7-20b again.

$$\frac{x^2-3x+46}{(x+3)(x^2-2x+17)} = \frac{A}{x+3} + \frac{Bx+C}{x^2-2x+17}$$

Heaviside method, step 2: Solve for A by multiplying equation 11.7-20b through by $(x+3)$ and evaluating it at $x=-3$.

$$\frac{x^2-3x+46}{x^2-2x+17} = A + \left.\frac{(Bx+C)(x+3)}{x^2-2x+17}\right]_{x=-3} = \frac{9+9+46}{9+6+17} = \frac{64}{32} = 2 \tag{11.7-25}$$

Evaluating this at $x=-3$ zeros out both B and C, and allows us to solve for A directly. We can see that $A=2$ from this equation alone.
Heaviside method, step 3: Put the partial fraction just solved for over the common denominator and subtract it from both sides. Now that we know that $A=2$, equation 11.7-20b becomes

$$\frac{x^2-3x+46}{(x+3)(x^2-2x+17)} = \frac{2}{x+3} + \frac{Bx+C}{x^2-2x+17} \tag{11.7-26a}$$

Putting that first partial fraction over a common denominator and subtracting will eliminate it from the right-hand side and will subtract $2(x^2-2x+17)$ from the numerator on the left.

$$\frac{-x^2+x+12}{(x+3)(x^2-2x+17)} = \frac{Bx+C}{x^2-2x+17} \tag{11.7-26b}$$

Heaviside method, step 4: Divide the common factor out of the numerator and denominator of the left side. Observe that the left side has a $(x+3)$ in its denominator, yet that factor does not appear in any denominator on the right. This means that $(x+3)$ must divide evenly into the left side's numerator. So we use polynomial long division to divide it out (again, this division should always yield no remainder, so if you get a nonzero remainder, you made a mistake somewhere).

$$
\begin{array}{r}
-x \quad +4 \\
x+3\,\overline{\big)\, -x^2 \quad +x+12} \\
\underline{x^2+3x \qquad\quad} \\
4x+12 \\
\underline{-4x-12} \\
0
\end{array}
$$

That leaves

$$\frac{-x+4}{x^2-2x+17} = \frac{Bx+C}{x^2-2x+17} \tag{11.7-26c}$$

It's pretty clear from this that $B = -1$ and $C = 4$. But if we had had more than one irreducible quadratic in the mix, we would have had to revert to the standard method at this point to resolve the remaining coefficients. Even in that case, solving the real-root coefficients and subtracting them all out in the manner we have just done means that when you do have to revert to the standard method, you will have fewer equations in fewer unknowns than if you had done the entire problem by the standard method. Of course the last step is the same no matter which method you use.

Food for Thought

If you look carefully into what we're doing when we apply Heaviside's method, you will find that we are evaluating a rational function precisely at a point where it is not defined (that is, where its denominator is zero). Of course this is a no-no and you would expect that it would invalidate the whole procedure. Can you make an argument based on limits that explains why Heaviside's method always works despite this shortcoming?

Another Application

If you are going into electrical or mechanical engineering, you will soon find that the method of partial fractions is used not only for finding integrals, but also for analyzing linear differential systems like resonant circuits and feedback-control systems. If you have an interest in these fields, consider taking some extra time now to learn the method of partial fractions well, so you won't have to waste time relearning it when you get to your circuits or linear systems courses. And by the way, would it surprise you to learn that the first person to use partial fractions to analyze linear differential systems was none other than Oliver Heaviside?

Exercises

1) Use the method of partial fractions (either standard or Heaviside, your choice) to find the indefinite integral

$$\int \frac{-6x^3+71x^2+283x-1614}{(x-4)(x-3)(x+5)(x+6)}\,dx$$

View standard method solution to problem 1
View Heaviside method solution to problem 1

2) Use the method of partial fractions (either standard or Heaviside, your choice) to find the indefinite integral

$$\int \frac{6x^3+55x^2+224x+1980}{(x-6)(x+5)(x^2+4x+40)}\,dx$$

View standard method solution to problem 2
View Heaviside method solution to problem 2

3) Use the method of partial fractions (either standard or Heaviside, your choice) to find the indefinite integral

$$\int \frac{11x^3-107x+108}{(x+7)(x-2)^3}\,dx$$

View standard method solution to problem 3
View Heaviside method solution to problem 3

4) Use the method of partial fractions (you have to use the standard method on this one because Heaviside won't work) to find the indefinite integral

$$\int \frac{3x^3+31x^2-55x-95}{(x^2-6x+25)(x^2+2x+5)}\,dx$$

View standard method solution to problem 4